JOURNAL OF APPROXIMATION THEORY **11**, 260–274 (1974)

Geometrical Characterizations for Nonlinear Uniform Approximation

DIETRICH BRAESS

Institut für Mathematik,
Ruhr-Universität, 463 Bochum, West Germany

1. INTRODUCTION

In the study of nonlinear approximation there has arisen the famous problem to characterize Chebyshev sets (i.e., sets for which there is always a unique best approximation) by geometrical properties [22]. When the best approximation is searched in subsets of smooth and strictly convex Banach spaces, then convexity is the dominating property. According to well-known results of Efimov and Stechkin [13] and of Vlasov [27] Chebyshev sets in these spaces are convex, provided that they are suns. Moreover, this additional assumption may be replaced by approximative compactness and it may be abandoned in finite-dimensional spaces. References for related problems are given in Refs. 12, 17 and 25.

On the other hand, convexity is neither necessary nor sufficient for uniqueness of best uniform approximation. This holds even in $\mathbb{R}^2$, as is shown by the examples in Fig. 1. Therefore, there has been a continuous search for those properties of nonlinear families of functions which are most essential. As a consequence, several properties and conditions (which are equivalent to being a sun or more restrictive) have been introduced by different authors. Since it is often difficult to understand the relationship between them, we list the properties known to us and verify their relationship before we prove that

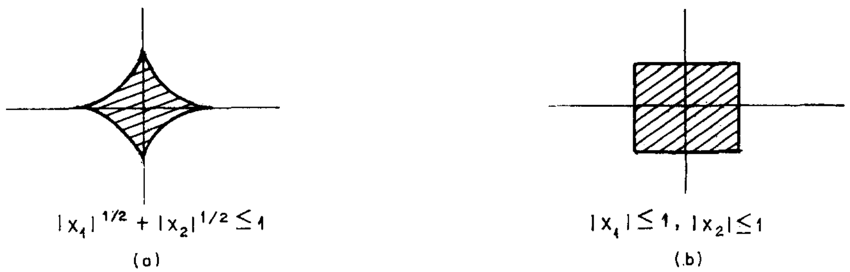

FIGURE 1.

total regularity dominates the uniqueness problem for uniform approximation. The results are very similar to those for smooth and strictly convex spaces, provided that convexity is replaced by total regularity.

2. List of Properties

Let Q be a compact set, and let the space of continuous, real valued functions $C(Q)$ be endowed with the uniform norm

$$\|f\| = \sup_{x \in Q} |f(x)|. \tag{2.1}$$

In particular, if Q is a finite set consisting of n points, then $C(Q)$ may be identified with $\mathbb{R}^n$ topologized according to (2.1). The set of extremal points of $f \in C(Q)$ will be denoted by $M[f]$:

$$M[f] = \{x \in Q;\ |f(x)| = \|f\|\}. \tag{2.2}$$

Let V be a nonvoid set of functions. When considering the properties we will present parameter-free definitions as far as possible. On the other hand, if we refer to tangent spaces, then we assume that V may be represented in the form

$$V = \{F[a] = F(a, x);\ a \in P\},$$

where P is an open subset in m-space and F possesses a Fréchet derivative with respect to a at each point $a \in P$. If a property requires that Q is an interval, we write $V \subset C(I)$.

The following properties are listed in Fig. 2.

(WH) A subspace $V \subset C(I)$ with dimension n satisfies the weak Haar condition, if each $v \in V$ has at most $n - 1$ changes of sign; i.e., there do not exist points $x_1 < x_2 < \cdots < x_{n+1}$ such that $v(x_i) \cdot v(x_{i+1}) < 0$, $i = 1, 2,..., n$ [16].

(B) V has the betweenness property, if given a pair $v_0, v_1 \in V$ there exists a mapping $[0, 1] \ni t \to v_t \in V$, such that for all $x \in Q$, $v_t(x)$ is either a strictly monotone continuous function of t or a constant [10]. We note that families with the betweenness property are boundedly connected [c.f. (HM)].

(R) V is regular, if given a pair $v, v_0 \in V$ and a closed set $A \subset Q$ with

$$\inf_{x \in A} |v(x) - v_0(x)| > 0, \tag{2.3}$$

the element v_0 is contained in the closure of the set

$$\{\tilde{v} \in V;\ (\tilde{v}(x) - v_0(x)) \cdot (v(x) - v_0(x)) > 0 \text{ for } x \in A\}. \tag{2.4}$$

[7.8]. The regularity is called closed sign property in Ref. 9.

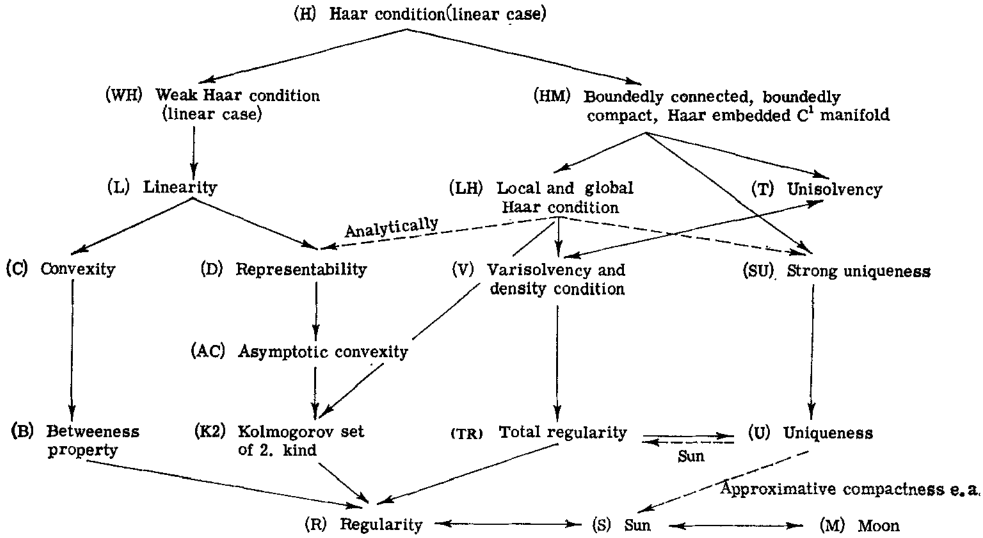

FIG. 2. Relationship of the properties.

(*D*) V satisfies the representation condition, if given a pair $v, v_0 \in V$ there exists a positive function $g \in C(Q)$ and a w in the tangent space at v_0 such that $v - v_0 = g \cdot w$ [18].

(*AC*) V is asymptotically convex, if given a pair $v, v_0 \in V$ there exists a positive function $g \in C(Q)$ and a mapping $[0, 1] \ni t \to v_t \in V$ such that

$$\|(1 - tg) \cdot v_0 + tgv - v_t\| = o(t) \tag{2.5}$$

as $t \to 0$. Originally, in Ref. 21 instead of (2.5) the relation

$$\|(1 - t \cdot g_t)\, v_0 + tg_t v - v_t\| = o(t) \tag{2.6}$$

is used, with the mapping $t \to g_t \in C(Q)$ being continuous. But (2.6) implies (2.5), if $g = g_0$ is inserted.

(*K2*) For $v_0 \in V$ the cone $\mathscr{R}[v_0, V]$ consists of the elements $g \in C(Q)$ satisfying the following. For every neighborhood U of g and for every $\epsilon > 0$ the set $\bigcup_{0<\eta<\epsilon} (v_0 + \eta U)$ intersects V. V is a Kolmogorov set of the second kind [8] if v_0 is a best approximation to f, whenever

$$\min_{x \in M[f-v_0]} (f(x) - v_0(x)) \cdot h(x) \leqslant 0 \tag{2.7}$$

holds for every $h \in \mathscr{R}[v_0, V]$.

(*HM*) An n-dimensional C^1-manifold $V \subset C(I)$ is Haar embedded, if the tangent space at each point is an n-dimensional Haar subspace. V is boundedly compact (boundedly connected, respectively) if the intersection of V with every open ball is relatively compact (connected, respectively) or it is empty [28, 29].

(LH) $V \subset C(I)$ satisfies the local Haar condition if the tangent space at each element v_0 is a Haar subspace. Let $d(v_0)$ be its dimension. V satisfies the global Haar condition, if for every $v \in V\backslash\{v_0\}$ the difference $v - v_0$ has at most $d(v_0) - 1$ zeros [21].

(T) $V \subset C(I)$ is an n-parameter family or unisolvent, if given (x_i, y_i), $(i = 1, 2, ..., n)$, $x_1 < x_2 < \cdots < x_n$ there is a unique $v \in V$, such that $v(x_i) = y_i$, $i = 1, 2, ..., n$ [23, 26].

(V) $V \subset C(I)$ is solvent of degree $m = m(v_0)$ at v_0, if given m distinct points x_i, $i = 1, 2, ..., m$ and $\epsilon > 0$ there is a $\delta = \delta(v_0, \epsilon, x_1 \cdots x_m) > 0$ such that $| y_i - v_0(x_i)| < \delta$ implies the existence of a $v \in V$, satisfying

$$v(x_i) = y_i, \qquad i = 1, 2, ..., m,$$

as well as $\| v - v_0 \| < \epsilon$. If, in addition, the difference $v - v_0$ has at most $m(v_0) - 1$ zeros for every $v \in V\backslash\{v_0\}$, then V is varisolvent [24]. V satisfies the density condition, if given $v_0 \in V$ and $\epsilon > 0$ there exists a $v \in V$ such that $0 < v(x) - v_0(x) < \epsilon$ (and $0 > v(x) - v_0(x) > -\epsilon$, respectively) [11]. Note that the definition of varisolvency given in Ref. 14 is more restrictive. As was pointed out in Ref. 3 varisolvent families may only be defined on sets that are homeomorphic to a subset of a circumference.

(TR) A pair v_1, v_0 of distinct elements of V is zero-sign compatible if given a closed subset Z of the zeros of $v_1 - v_0$ and $s \in C(Z)$ taking the values $+1$ and -1, there exists a $v \in V$ such that $\operatorname{sign}(v(x) - v_0(x)) = s(x)$ for $x \in Z$ [9]. If all pairs of distinct elements of a regular set V are zero-sign compatible, V is called totally regular.

(U) V is a uniqueness set, if for each $f \in C(Q)$ there is at most one best approximation to f in V.

(SU) V satisfies a strong uniqueness condition, if given $f \in C(Q)$ and a best approximation v_0 to f in V there is a constant $c > 0$ such that

$$\| f - v \| \geqslant \| f - v_0 \| + c \| v - v_0 \|$$

for $v \in V$.

(S) V is a sun if whenever $v_0 \in V$ is a best approximation to $f \in C(Q)$, then v_0 is also a best approximation to $v_0 + \lambda(f - v_0)$ for all $\lambda \geqslant 0$ [12].

(M) $K(v_0, f) = \bigcup_{\lambda > 0} B(v_0 + \lambda(f - v_0), \lambda \| f - v_0 \|)$ with $B(f, r)$ denoting the open ball with radius r centered at f. V is a moon if $v_0 \in V$, $f \in C(Q)$ and $V \cap K(v_0, f) \neq \varnothing$ implies that the closure of $V \cap K(v_0, f)$ contains v_0, [1].

3. EXAMPLES

Before we verify the relationship of the properties shown in Fig. 2 we give some examples and note some simple features.

Observe that apart from (*K*2), (*SU*), (*U*) (*M*) and (*S*) the families are characterized by intrinsic properties, i.e., the definitions do not refer to the approximation of elements. Moreover, the graph may nearly be divided into two branches. The left-hand branch is dominated by conditions like convexity and the right one by varisolvency. This seems quite natural since the Haar subspaces are the only families that are both convex and varisolvent.

We note that (*D*), (*AC*), (*K*2), (*R*), (*LH*), (*V*) and (*TR*) are hereditary, i.e., whenever the family V has the property, this also holds for every nonvoid subfamily $V \cap U$ provided that U is open in $C(Q)$.

Now we list some examples.

(*WC*) Spline polynomials. Let k be an arbitrary positive integer and define the functions $(x)_+^k = x^k$ if $x \geqslant 0$, $= 0$ otherwise. Choose knots $-1 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. The linear subspace spanned by $1, x \cdots x^k$, $(x - \xi_1)_+^k \cdots (x - \xi_n)_+^k$ satisfies the weak Haar condition [16].

(*C*) The set of nondecreasing polynomials of degree $\leqslant n$ is convex [20].

(*B*) Let Q be a set with two points. Set

$$V = \{v \in \mathbb{R}^2, v = (\alpha, \alpha^3), \alpha \in \mathbb{R}\}. \tag{3.1}$$

Obviously, V has the betweenness property. As was pointed out in Ref. 8 (p. 374) this set is not a Kolmogorov set of the second kind.

(*D*) The proper sums of exponentials

$$\left\{\sum_{j=1}^{n} \alpha_j e^{t_j x};\ \alpha_j, t_j \in \mathbb{R}, j = 1, 2 \cdots n\right\} \tag{3.2}$$

satisfy the representation property [18, p. 286].

(*AC*) Let V_1 and V_2 be convex sets in $C(Q)$. Then the set of rational functions $V = \{v = v_1/v_2;\ v_2(x) > 0,\ v_1 \in V_1,\ v_2 \in V_2\}$ is asymptotically convex provided that it is not empty [21, p. 305].

(*K*2) Let $P = \bigcup_{k=1}^{\infty} \{\alpha \in \mathbb{R}; 4^{-k} < \alpha < 2 \cdot 4^{-k}\} \cup \{0\}$. Then the set of constant functions $V = \{v; v(x) = \alpha,\ \alpha \in P\}$ is a Kolmogorov set of the second kind which is not asymptotically convex.

(*HM*) Let $p \in C^1(\mathbb{R})$ with $p' \neq 0$. If H is a Haar subspace, then $V = \{v(x) = p(h(x)); h \in H\}$ is a C^1 manifold with the additional properties [28, p. 369].

(*T*) The set of functions

$$V = \{v(x);\ v(x) = -cx + d \text{ or } v(x) = ce^x + d,\ c, d \in \mathbb{R},\ c \leqslant 0\}$$

is a two-parameter family [26].

(*LH*) The γ-polynomials

$$\left\{\sum_{j=1}^{n} \alpha_j \gamma(t_j\,; x);\ \alpha_j \in \mathbb{R},\ t_j \in T, j = 1, 2 \cdots n\right\} \tag{3.3}$$

satisfy the local and global Haar condition provided that the kernel $\gamma \in C(I \times T)$ is extended sign-regular [4]. A well-known example for γ-polynomials are the exponentials (3.2).

(*V*) The γ-polynomials are varisolvent if the kernel is only strictly sign-regular and the derivatives $\partial\gamma(t, x)/\partial t$ do not exist [4, Theorem 4.1]. Moreover, if the exponentials are generated by the kernel $\gamma(t, x) = \exp(t^3 x)$ and not by $\exp(tx)$, then the family (3.3) has the property (*V*) but not (*LH*). Referring to exponentials again we emphasize that the closure of a varisolvent family need not be varisolvent and not even regular.

(*TR*) The subset of exponentials with non-negative coefficients

$$V = \left\{\sum_{j=1}^{n} \alpha_j e^{t_j x};\ \alpha_j \geqslant 0,\ t_j \in \mathbb{R}\right\}$$

is not varisolvent but it is totally regular. Since best approximations may be characterized by alternants [2] the set is a sun and regular. When v_0 has exactly k positive α_j, $1 \leqslant j \leqslant n$, then $v - v_0$ has at most $2k$ zeros, whenever $v \in V$ and V is solvent of degree $2k$ at v_0. This implies zero-sign compatibility. Finally, we note that the set (3.1) is totally regular. Hence, totally regular sets are not always Kolmogorov sets of the second kind.

4. Comparison of Properties

In this section we will verify the relationship between the properties, leaving only the restricted implications $(U) \Rightarrow (S)$ and $(R) \Rightarrow (B)$ to later considerations. We will confine ourselves to give a reference as far as possible.

$(H) \Rightarrow (WH) \Rightarrow (L) \Rightarrow (C) \Rightarrow (B) \Rightarrow (R)$. Obvious.

$(C) \Rightarrow (AC)$. Set $v_t = v_0 + t(v - v_0)$ and $g \equiv 1$.

$(AC) \Rightarrow (K2)$. Let v_0 satisfy the local Kolmogorov criterion (2.7) and let v be a distinct point in V. Choose g and v_t according to the definition of asymptotic convexity and set $h(x) = g(x)(v(x) - v_0(x))$. By virtue of (2.5) we have

$$v_t = v_0 + t(h + k_t) \tag{4.1}$$

with $\| k_t \| = o(t)$ as $t \to 0$. If U is an arbitrary neighborhood of h, then $h + k_t \in U$ holds for sufficiently small t. Hence, $h \in \mathscr{R}(v_0, V)$ and

$$\min_{M[f-v_0]} (f(x) - v_0(x)) \cdot h(x) \leqslant 0.$$

From $g(x) > 0$ it follows that

$$\min_{M[f-v_0]} (f(x) - v_0(x)) \cdot (v(x) - v_0(x)) \leqslant 0.$$

Since this inequality holds for each $v \in V$, it follows that v_0 is a best approximation and that (2.7) is a sufficient condition. This result improves the statement that convex sets are Kolmogorov sets of the second kind [8, Theorem 3].

$(K2) \Rightarrow (K1)$. Reference 8.

$(L) \Rightarrow (D)$. Obvious.

$(D) \Rightarrow (AC)$. Theorem 4 in Ref. 18.

$(H) \Rightarrow (HM)$. Obvious.

$(HM) \Rightarrow (LH)$. Haar embedded manifolds satisfy the local Haar condition by definition. Assume that the global Haar condition is violated and that the difference $v_1 - v_0$ has n zeros $(v_1 \neq v_0)$. Then by standard arguments [24, p. 299] a function v_2 is constructed such that $v_2 - v_0$ has n changes of sign. Choose $\xi_1 < \xi_2 < \cdots < \xi_{n+1}$ such that

$$(v_2 - v_0)(\xi_i) \cdot (v_2 - v_0)(\xi_{i+1}) < 0, \qquad i = 1, 2 \cdots n,$$

and let $s \in C(I)$, $\| s \| = 1$ satisfy

$$s(\xi_i) = \operatorname{sign}(v_2 - v_0)(\xi_i), \qquad i = 1, 2 \cdots n + 1,$$
$$s(x) \cdot (v_2 - v_0)(x) \geqslant 0, \qquad x \in I.$$

Then v_0 is not a unique best approximation to $f = v_0 + \| v_2 - v_0 \| \cdot s$, contradicting the assertion that $f - v_0$ alternates $n + 1$ times.

$(HM) \Rightarrow (T)$. If n points (x_i, y_i), $i = 1, 2 \cdots n$ are given, it follows from bounded compactness that the infinum of the function

$$\max_{1 \leqslant i \leqslant n} | v(x_i) - y_i |$$

is attained at some elements v_0 in V. Assume that the minimal value is not zero. Then a function $f \in C(I)$ satisfying $f(x_i) = y_i$, $i = 1, 2 \cdots n$ is easily constructed, to which v_0 is not a best approximation in V. This is a contradiction. Since V satisfies the global Haar condition, the solution of the interpolation problem is unique.

$(HM) \Rightarrow (SU)$. Let v_0 be the best approximation to f. To prove a strong uniqueness relation we consider three regions. By virtue of Lemma 9 in Ref. 28 there is an $r > 0$ and a $c_1 > 0$ such that $\| v - v_0 \| < r$, $v \in V$ implies

$$\| f - v \| \geqslant \| f - v_0 \| + c_1 \| v - v_0 \|.$$

From uniqueness and bounded compactness we conclude that in the region $r \leqslant \| v - v_0 \| \leqslant 3 \| f - v_0 \|$, $v \in V$,

$$\inf\{\| f - v \|\} = \alpha > \| f - v_0 \|.$$

Hence, there the inequality

$$\| f - v \| \geqslant \| f - v_0 \| + c_2 \| v - v_0 \|$$

holds with $c_2 = (1/3)(\alpha - \| f - v_0 \|)/\| f - v_0 \|$. Finally, if $\| v - v_0 \| > 3 \| f - v_0 \|$ is valid, the triangle inequality yields

$$\| f - v \| \geqslant \| v - v_0 \| - \| f - v_0 \| \geqslant \| f - v_0 \| + (1/3) \| v - v_0 \|.$$

This proves strong uniqueness.

$(LH) \Rightarrow (V)$. By virtue of the implicit function theorem the local Haar condition implies solvency of degree $d(v_0)$. Combining this with the global Haar condition yields varisolvency. Since $v \in V$ is not a best approximation to $f = v + (\epsilon/2) \cdot 1$ and $f = v - (\epsilon/2) \cdot 1$, respectively, we obtain the density condition.

$(LH) \Rightarrow (K2)$. Theorem 7 in Ref. 8.

$(LH) \Rightarrow (D)$ provided that the elements of V and of the tangent spaces are analytic functions and that zeros may be counted with their multiplicities: Theorem 7 in Ref. 8.

$(LH) \Rightarrow (SU)$ holds with the following restriction. If V satisfies the local and global Haar condition, and if f has a best approximation v_0 such that $d(v_0)$ is maximal in V, then v_0 satisfies a strong uniqueness condition. The proof proceeds similarly to that of $(HM) \Rightarrow (SU)$.

$(T) \Rightarrow (V)$. By definition unisolvency implies varisolvency. The density property follows from the fact that the degree of solvency is constant [5].

$(V) \Rightarrow (TR)$. Since the best approximation may be characterized by alternants, it follows that the sets with property (V) are suns and regular. The zero-sign compatibility is a consequence of the fact that the degree of solvency is greater than the number of zeros.

$(TR) \Rightarrow (R)$. By definition.

$(SU) \Rightarrow (U)$. Obvious.

$(TR) \Rightarrow (U)$ and $(U) \Rightarrow (TR)$ provided V is a sun. [9, Theorem 5].

$(R) \Leftrightarrow (S) \Leftrightarrow (M)$. This is a consequence of a more general result.

THEOREM 4.1. *For every nonvoid set* $V \subset C(Q)$ *the following are equivalent.*

(a) *V is regular.*

(b) *V is a Kolmogorov set of the first kind, i.e., if $v_0 \in V$ is a best approximation to f in V, then for every $v \in V$*

$$\inf_{x \in M[f-v_0]} (f(x) - v_0(x))(v(x) - v_0(x)) \leqslant 0.$$

(c) *V is a sun.*

(d) *V is a moon.*

(e) *Every local best approximation in V is a best approximation.*

(f) *If v_0 is a best approximation to f in V, then for every $v \in V$ the element v_0 is also a best approximation to f in the convex hull of v and v_0 (which may be interpreted as the straight line between v_0 and v).*

The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) *and* (c) $\Leftrightarrow$ (f) *were proved in Refs.* 8 *and* 6, *respectively. Moreover* (c) $\Leftrightarrow$ (d) *is a consequence of Corollary* 2.9 *and Corollary* 4.2 *in Ref.* 1. *Finally, by virtue of Theorems* 3 *and* 4 *in Ref.* 9 *we have* (e) $\Leftrightarrow$ (a).

5. CHARACTERIZATION OF CHEBYSHEV SETS

By collecting properties after some redefinitions we established for the uniform approximation that a sun is a uniqueness set if and only if it is totally regular. Hence, we have a characterization by an intrinsic property under an additional assumption which is less restrictive than the assumptions in Refs. 7 and 19. (Referring to the approximation by positive sums of exponentials [the example to (TR) in Section 3] it seems plausible that a general uniqueness criterion will not only incorporate numbers of zeros). Furthermore, observe that the uniqueness theorem in Ref. 15 does not use only intrinsic properties.

Now the situation is very similar to the approximation in smooth and uniformly convex spaces, where suns are uniqueness sets if and only if they are convex [13, 27]. In those spaces Chebyshev sets are known to be suns provided that they are approximatively compact [25]. We will establish an analogous result for the uniform norm.

THEOREM 5.1. *Let $V \subset C(Q)$ be a Chebyshev set. Assume that one of the following conditions holds.*

(a) *V is approximately compact.*

(b) *The operator π_v: $C(Q) \to V$ which sends $f \in C(Q)$ to its best approximation, is continuous.*

(c) $\dim C(Q) < \infty$.

Then V is a sun.

Proof. Suppose that V is not regular. Then there is a pair v, $v_0 \in V$, a closed nonvoid set $A \subset Q$ satisfying (2.3) and a real number $\lambda > 0$, such that

$$\{\tilde{v} \in V; \| \tilde{v} - v_0 \| < \lambda, (\tilde{v}(x) - v_0(x))(v(x) - v_0(x)) > 0 \text{ for } x \in A\} = \phi. \tag{5.1}$$

By virtue of Uryson's extension theorem there is a function $\psi \in C(Q)$

$$\psi(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } v(x) = v_0(x), \\ 0 < \psi < 1, & \text{otherwise.} \end{cases}$$

Then $\varphi(x) = \psi(x) \cdot \operatorname{sign}[v(x) - v_0(x)]$ defines a continuous function. For $\alpha \geqslant 0$, let

$$f_\alpha = v_0 + \alpha \cdot \varphi.$$

Since A is nonvoid, it follows that $\| \varphi \| = 1$ and

$$\| f_\alpha - v_0 \| = \alpha. \tag{5.2}$$

From (5.1) we conclude that $v_0 = \pi_v(f_\alpha)$, $0 \leqslant \alpha < \lambda/2$. On the other hand we have with $\beta = \| v - v_0 \|$,

$$\begin{aligned} | f_\beta(x) - v(x) | &= | v_0(x) + \beta\varphi(x) - v(x) | \\ &= \big| \, | v(x) - v_0(x) | - \beta\psi(x) \big|. \end{aligned} \tag{5.3}$$

Since the right side of (5.3) is bounded by the difference of two non-negative terms not exceeding β, it follows that

$$\| f_\beta - v \| \leqslant \beta = \| f_\beta - v_0 \|. \tag{5.4}$$

Uniqueness implies $v_0 \neq \pi_v(f_\beta)$. Set

$$\gamma = \sup\{\alpha \in \mathbb{R}; \pi_v(f_\alpha) = v_0\}. \tag{5.5}$$

As is well known, the set on the right-hand side of (5.5) is connected and closed. This yields $v_0 = \pi_v(f_\gamma)$. From (5.4) we know that $\gamma < \beta$ holds. By standard arguments the sequence $v_\rho = \pi_v(f_{\gamma+1/\rho})$, $\rho = 1, 2, 3,\ldots$ is a minimal sequence for f_γ. By virtue of $v_\rho \neq v_0$ and (5.1) we obtain

$$\| v_\rho - v_0 \| \geqslant \lambda. \tag{5.6}$$

If V is approximately compact, a subsequence of $\{v_\rho\}$ converges to a best approximation v^* of f_γ, satisfying $\| v^* - v_0 \| \geqslant \lambda$. This contradicts uniqueness.

If π_v is continuous, then (5.6) establishes a contradiction.

Every Chebyshev set is closed. It is approximately compact in spaces of finite dimension. Hence, the proof for (c) reduces to (a). ∎

Since approximately compact sets are existence sets, we have the following:

COROLLARY 5.2. *An approximatively compact set in $C(Q)$ is a Chebyshev set, if and only if it is totally regular.*

In the finite-dimensional case even fewer assumptions are necessary.

COROLLARY 5.3. *Let Q be a finite set. A nonvoid set $V \subset C(Q)$ is a Chebyshev set, if and only if it is closed and totally regular.*

6. REGULAR SETS IN $\mathbb{R}^n$

In this section we will prove that closed, regular sets have a property which almost coincides with the betweenness property, provided that Q is finite and consists of n points. To be more precise, we will prove the following.

THEOREM 6.1. *Let V be a regular, closed set in $\mathbb{R}^n$. Then given $u, w \in V$, there exists a continuous arc v_t, $0 \leqslant t \leqslant 1$ from u to w in V such that all coordinates v_t^m are monotone functions of t. The functions v_t^m are strictly monotone, provided that*

$$u^k \neq w^k, \qquad k = 1, 2 \cdots n. \tag{6.1}$$

For convenience, we assume $u^k \leqslant w^k$, $k = 1, 2 \cdots n$. We use the notation

$$[u, w] = \{v \in \mathbb{R}^n;\, u^k \leqslant v^k \leqslant w^k\}$$

and denote the interior of this set by int$[u, w]$. The proof of the theorem is prepared by a lemma.

LEMMA 6.2. *Let V be a regular, closed set in $\mathbb{R}^n$. Let $1 \leqslant m \leqslant n$. Assume that $u, w \in V$ and $u^m < \alpha < w^m$.*

(i) *There is a* $\bar{v} \in V \cap [u, w]$ *satisfying* $\bar{v}^m = \alpha$.

(ii) *There is a* $\bar{v} \in V \cap \operatorname{int}[u, w]$ *satisfying* $\bar{v}^m = \alpha$ *provided that the strict inequalities* (6.1) *hold.*

Proof of Lemma 6.2. Given $\epsilon > 0$, consider the set

$$\{v \in V; u^k - \epsilon \leqslant v^k \leqslant w^k + \epsilon, k = 1, 2 \cdots n, v^m \leqslant \alpha\}.$$

Since V is closed, this set is compact and the function $\varphi(v) = v^m$ attains its maximum in this set at a point called v_ϵ. We have $v_\epsilon^m = \alpha$, since $v_\epsilon^m < \alpha$ leads to a contradiction to regularity when applied to the pair v_ϵ, w. Since ϵ is arbitrary, the proof of (i) is completed by compactness arguments.

If (6.1) holds and V is regular we may chose $\tilde{u}$, $\tilde{w} \in V$ such that

$$u^k < \tilde{u}^k \leqslant \tilde{w}^k < w^k, \qquad k = 1, 2 \cdots n,$$
$$\tilde{u}^m < \alpha < \tilde{w}^m.$$

By applying (i) to the pair $\tilde{u}$, $\tilde{w}$ we obtain (ii). ∎

Proof of Theorem 6.1. Starting with $B_0 = \{u, w\}$, we construct sets B_ν with $2^\nu + 1$ elements, $\nu = 1, 2, 3...$ satisfying

$$B_0 \subset B_1 \subset \cdots \subset B_\nu \subset B_{\nu+1} \subset \cdots \subset V \cap [u, w]. \tag{6.2}$$

Moreover, the elements of each B_ν can be ordered such that with increasing index all coordinates are nondecreasing [strictly increasing, respectively, provided that (6.1) holds].

Assume that B_ν has been constructed. Let $B_\nu = \{v_\rho; \rho = 0, 1 \cdots 2^\nu\}$, where the elements are ordered as stated above. Now we choose an integer m, $1 \leqslant m \leqslant n$, satisfying

$$m - 1 \equiv \nu (\operatorname{mod} n). \tag{6.3}$$

By applying Lemma 6.2 to the pairs $v_{\rho-1}$, v_ρ, $\rho = 1, 2 \cdots 2^\nu$ we obtain 2^ν points $\bar{v}_\rho \in V$ with

$$\bar{v}_\rho^m = (1/2)(v_{\rho-1}^m + v_\rho^m)$$

and

$$v_{\rho-1}^k \leqslant \bar{v}_\rho^k \leqslant v_\rho^k, \quad (v_{\rho-1}^k < \bar{v}_\rho^k < v_\rho^k, \text{respectively}), \quad k = 1, 2 \cdots n \tag{6.4}$$

By adding these 2^ν points to B_ν, the set $B_{\nu+1}$ is defined. Obviously, $B_{\nu+1}$ has the properties stated above. Observe that by virtue of (6.3) the distance between successive points is reduced to less than a half, whenever n steps are performed. Hence, the distance between successive points in $B_{n \cdot \mu}$ is not greater than

$$2^{-\mu} \| w - u \|. \tag{6.5}$$

Set $B_\infty = \bigcup_{\nu=1}^{\infty} B_\nu$. We use the notation B for the closure of B_∞ in $\mathbb{R}^n$. The mapping

$$\varphi: B \to [0, 1]$$

$$\varphi(v) = \frac{\sum_{k=1}^{n} (v^k - u^k)}{\sum_{k=1}^{n} (w^k - u^k)}$$

is continuous. By (6.5) we conclude that $\varphi(B)$ is dense in $[0, 1]$. Since B and $\varphi(B)$ are compact, it follows that $\varphi(B) = [0, 1]$. To prove that φ is injective, let $v_1, v_2 \in B$ and $v_1 \neq v_2$. For convenience we assume that $v_1^m < v_2^m$ for some $m \leqslant n$. By the definition of B and (6.5) there exist points $\tilde{v}_1, \tilde{v}_2 \in B$ satisfying

$$v_1^m \leqslant \tilde{v}_1^m < \tilde{v}_2^m \leqslant v_2^m.$$

The construction assures

$$\tilde{v}_1^k \leqslant \tilde{v}_2^k \qquad (v_1^k < v_2^k, \text{ respectively}), \qquad k = 1, 2 \cdots n.$$

Since v_1 and v_2 are cluster points of elements of $B \cap [u, \tilde{v}_1]$ and $B \cap [\tilde{v}_2, w]$ we have

$$v_1^k \leqslant v_2^k \qquad (v_1^k < v_2^k, \text{ respectively}), \qquad k = 1, 2 \cdots n,$$

and $\varphi(v_1) < \varphi(v_2)$. Since φ is a bijective mapping of a compact set, φ^{-1} is continuous. Moreover, the preceding discussion establishes the monotonicity of the coordinates. ∎

From Theorem 6.1 we know that every regular and closed set in $\mathbb{R}^n$ is connected. Even the intersection with every closed ball is connected. As a consequence we have

COROLLARY 6.3. *Let $V \subset R^n$ be a regular existence set. Then for every $f \in \mathbb{R}^n$ the set of the best approximations in V is connected.*

As is shown in Fig. 3, Theorem 6.1 yields a geometrical structure that is comparable with convexity. If $u, w \in V \subset \mathbb{R}^2$, convexity requires that the straight line between u and w is contained in V, whereas regularity requires

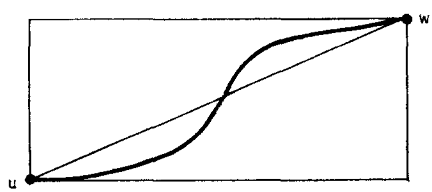

FIG. 3. Convexity and regularity.

only the existence of an arc from u to w in V, which proceeds within the rectangle $[u, w]$.

The difference between regularity and total regularity becomes apparent when a pair $u, w \in V \subset \mathbb{R}^2$ is considered whose second coordinates coincide (Fig. 4). If V is totally regular, then in every neighborhood of the straight line $[u, w]$ there is an arc from u to w above the line and an arc below it. On the other hand, regularity only implies that the straight line belongs to V.

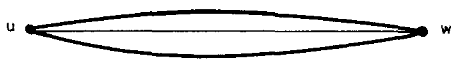

FIG. 4. Total regularity.

Note Added in Proof. Recently Dunham [31] has given an example of a Chebyshev set which is not a sun. If we would drop the word "continuous" in the definition of the betweenness property, then the set of the example would have that property. This fact also illustrates that continuity is essential for the proof of the implication $(B) \Rightarrow (R)$.

As was pointed out by Brosowski and Deutsch [30], the hypothesis in Theorem 5.1 may be relaxed.

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